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A NEW ALGORITHM FOR BIRKHOFF NORMALIZATION OF HAMILTONIAN SYSTEMS†

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Existing multidimensional recurrent algorithms for the normalization of Hamiltonian systems are replaced by a one-dimensional algorithm, that is, the whole procedure of computing the normal form is reduced to a single recurrent formula. This simplification is achieved by using the ring of asymptotic approximations of fixed order rather than Taylor series in powers of a formal parameter. © 1997 Elsevier Science Ltd. All rights reserved.

The reduction of systems of ordinary differential equations to Poincaré normal form in the neighbourhood of a singular point is simplified if the equations are in Hamiltonian form, since one can then apply the necessary transformation to the Hamiltonian rather than to the system itself. The definition of the normal form of a Hamiltonian and the first algorithm for reduction to normal form were proposed by Birkhoff [1]. Subsequently, the algorithm was appreciably improved by changing from the generating function of the required canonical transformation to a generating Hamiltonian, using Lie series [2, 3]. The next step was the development of closed recurrent algorithms that enabled the analytical computations for reducing Hamiltonians to normal form to be programmed for computers [4].‡

The approach employed below was used previously to simplify algorithms for normalizing differential systems in Poincaré's sense and the algorithm of the Krylov–Bogolyubov method [5].

We will consider Hamiltonian systems described by an autonomous analytic Hamiltonian

$$H(q, p) = H_0(q, p) + H_*(q, p) \tag{1}$$

where $H_0(q,p)$ is the quadratic part of the Hamiltonian, defining the linear part of the system, and $H_*(q, p)$ is a finite or infinite polynomial with no terms of less than third degree. In the theory of quasi-linear systems $H_0(q, p)$ is referred to as the unperturbed Hamiltonian and $H_*(q, p)$ as the perturbation.

When defining the concept of "Birkhoff normal form" and reducing to it, one usually replaces the original generalized momenta p by complex combinations of them: $x = p + iq, y = p - iq$. These relations may be regarded as a canonical change of variables $(q, p) \rightarrow (x, y)$ with valence $2i$.

Retaining the old notation for functions in the new variables, we shall assume that the valence has also been taken into consideration. Thus, we will assume from the start that the Hamiltonian is written as a function of x and y

$$H(x, y) = H_0(x, y) + H_*(x, y) \tag{2}$$

It is well known that a suitable change of variables will reduce the quadratic part of the Hamiltonian of a conservative oscillating system to a simpler form, namely

$$H_0 = \frac{1}{2} \sum_{k=1}^{n} \lambda_k (q_k^2 + p_k^2), \quad H_0(x, y) = i \sum_{k=1}^{n} \lambda_k x_k y_k$$

One can formulate the analogous problem of reducing the perturbed part of the Hamiltonian $H_*(x, y)$ to a simpler form. When that is done the changes of variables solving the problem must satisfy three conditions: they must be canonical; they must leave the quadratic part of the Hamiltonian, i.e. H_0, unchanged; and, finally, they must be polynomial functions. The third condition implies that the required transformations can have no singularities at zero.

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‡See also MARKEYEV A. P. and SOKOL'SKII A. G., Some computational algorithms for the normalization of Hamiltonian systems. Preprint No. 31, Moscow, Inst. Prikladnoi Matematiki, Akad. Nauk SSSR, 1976.

A Hamiltonian that cannot be simplified by any polynomial transformation is said to be in Birkhoff normal form (BNF).

The general structure of BNF is easily established. Suppose it is necessary to simplify $H_*(x, y)$ by eliminating a term

$$x_1^{l_1} \ldots x_n^{l_n} y_1^{s_1} \ldots y_n^{s_n}$$

This may be done by a canonical transformation $(x, y) \to (u, v)$ with a generating function $S(x, v)$ containing exactly the same term

$$S = \sum_k x_k v_k + h x_1^{l_1} \ldots x_n^{l_n} v_1^{s_1} \ldots v_n^{s_n}$$

(here the part $\sum_k x_k v_k$ of the generating function generates the identical transformation).

Applying the changes of variables

$$y = \frac{\partial S}{\partial x} = v + h \frac{\partial}{\partial x} (x^l v^s) = v + h \frac{\partial}{\partial u} (u^l v^s) + \ldots$$

$$u = \frac{\partial S}{\partial v} = x + h \frac{\partial}{\partial v} (x^l v^s) = x + h \frac{\partial}{\partial y} (x^l y^s) + \ldots \qquad (3)$$

for which we also write the inverse expressions

$$v = y - h \frac{\partial}{\partial x} (x^l y^s) + \ldots, \quad x = u - h \frac{\partial}{\partial v} (u^l v^s) + \ldots$$

we obtain the following expression for the Hamiltonian (2)

$$\tilde{H} = i \sum_{k=1}^{n} \lambda_k u_k v_k + ih \sum_{k=1}^{n} \lambda_k \left(u_k \frac{\partial}{\partial u_k} - v_k \frac{\partial}{\partial v_k} \right) u^l v^s + H_*$$

The term proportional to h has the same structure as the term that is to be suppressed in all cases, except when the expression $u^l v^s = u_1^{l_1} \ldots u_n^{l_n} v_1^{s_1} \ldots v_n^{s_n}$ is a first integral of the system of linear differential equations generated by the Hamiltonian H_0.

Thus, using polynomial transformations, one can suppress all terms in the non-linear part of H_* except the first integrals of the linear part.

The system with Hamiltonian H_0

$$\dot{u}_k = i\lambda_k u_k, \quad \dot{v}_k = -i\lambda_k v_k \quad (k = 1, \ldots, n)$$

has $2n - 1$ independent first integrals $G_1(u, v), \ldots, G_{2n-1}(u, v)$. The first n of them are polynomial in form

$$G_k = u_k v_k \ (k = 1, \ldots, n) \qquad (4)$$

Among the other first integrals, there are none in polynomial form if the system is non-resonant, i.e.

$$k_1 \lambda_1 + \ldots + k_n \lambda_n \neq 0 \quad (k_1^2 + \ldots + k_n^2 \neq 0) \qquad (5)$$

for any integers $k_1, \ldots, k_n$.

But if there are resonances, the number of independent polynomial first integrals is greater than n by the number of resonance relations of type (5).

For example, if there is a resonance $\lambda_1 = 3\lambda_2$, the polynomial integral added to (4) will be $G_{n-1} = u_1 v_2^3$. In this example the BNF may depend only on the arguments $u_1 v_1, \ldots, u_n v_n, u_1 v_2^3$.

Definition. A Hamiltonian has BNF when it depends only on polynomial first integrals (of invariants) of the unperturbed part, i.e. $H = H_0 + H_*$ has BNF if $\{H_0, H_*\} = 0$, (where $\{H_0, H_*\}$ is the Poisson bracket).

The algorithm described for the reduction of a Hamiltonian to BNF may be replaced by a much simpler one if the required canonical transformations are expressed not in terms of the generating function (3) but in terms of the Hamiltonian of some auxiliary system, whose phase flow is defined by the one-parameter group of transformation used to normalize the initial Hamiltonian.

Suppose that the Hamiltonian of this auxiliary system is $Q(x, y)$. That means that the equations

$$dx/d\tau = \partial Q/\partial y, \quad dy/d\tau = -\partial Q/\partial x$$

with initial data $x(0) = u, y(0) = v$ define the required transformations by the formulae

$$x = x(u, v, \tau), \quad y = y(u, v, \tau) \tag{6}$$

where the latter constitute a solution of this Cauchy initial-value problem.

The transformations (6) and their inverses may be written in terms of the following Lie series [5, 6]

$$x = u + \tau\{u, Q\} + \frac{\tau^2}{2!}\{\{u, Q\}, Q\} + \ldots, \quad y = v + \tau\{v, Q\} + \frac{\tau^2}{2!}\{\{v, Q\}, Q\} + \ldots$$

$$u = x - \tau\{x, Q\} + \frac{\tau^2}{2!}\{\{x, Q\}, Q\} + \ldots, \quad v = y - \tau\{y, Q\} + \frac{\tau^2}{2!}\{\{y, Q\}, Q\} + \ldots$$

In the case of the direct transformations, the symbols for the arguments x and y in the function Q are replaced by the symbols u and v. The function Q itself is unchanged.

The transformed Hamiltonian is also related to the original one by a Lie series

$$\tilde{H}(u, v) = H(u, v) + \tau\{H, Q\} + \frac{\tau^2}{2!}\{\{H, Q\}, Q\} + \ldots \tag{7}$$

This formula is fundamental for constructing the algorithm for the reduction of Hamiltonians to BNF.

We will formalize the criterion determining the order of the non-linear terms by scaling $x \to \varepsilon x$, $y \to \varepsilon y$. Then the Hamiltonian (2), taking the valence of this transformation into consideration, may be written as

$$H(x, y) = H_0(x, y) + \frac{1}{\varepsilon^2}H_*(\varepsilon x, \varepsilon y) \equiv H_0(x, y) + \varepsilon H_*(x, y, \varepsilon)$$

We introduce notation for the asymptotic forms of the Hamiltonian

$$H_k(x, y) = H + O(\varepsilon^{k+1}) \tag{8}$$

that is, H_k is any function that differs from the exact Hamiltonian by terms of higher order of smallness than ε^k.

The Hamiltonian of the auxiliary generating system will also be sought in this asymptotic form

$$Q_k = Q + O(\varepsilon^k)$$

Using series (7), one can find the relation between the asymptotic approximations of the BNF and those of the initial Hamiltonian and the Hamiltonian of the transformation group. Setting $\varepsilon = \tau$, we obtain

$$\tilde{H}_0(u, v) = H_0(u, v)$$

$$\tilde{H}_1(u, v) = H_1(u, v) + \tau\{H_0, Q_0\} \tag{9}$$

$$\tilde{H}_2(u, v) = H_2(u, v) + \tau\{H_1, Q_1\} + \frac{\tau^2}{2!}\{\{H_0, Q_0\}, Q_0\}$$

. . . .

$$\tilde{H}_k(u,v) = H_k(u,v) + \tau\{H_{k-1},Q_{k-1}\} + \ldots + \frac{\tau^k}{k!}\underbrace{\{\ldots\{H_0,Q_0\},Q_0\}\ldots\}}_{k \text{ times}}$$

These relations are derived using the properties of the ring of asymptotic approximations.

In the expression for $\tilde{H}_k(u, v)$, let us express the Poisson bracket $\{H_{k-1}, Q_{k-1}\}$ in an asymptotically equivalent form

$$\{H_{k-1},Q_{k-1}\} = \{H_0,Q_{k-1}\} + \{H_{k-1}-H_0,Q_{k-1}\} = \{H_0,Q_{k-1}\} + \{H_{k-1}-H_0,Q_{k-2}\}.$$

The two forms are indeed equivalent because $H_{k-1} - H_0 \sim \tau$.

Taking this representation into consideration, we introduce new notation

$$L_1 = H_1(u, v)$$

$$L_2 = \tau\{H_1 - H_0, Q_0\} + \frac{\tau^2}{2!}\{\{H_0,Q_0\},Q_0\} + H_2(u,v) \tag{10}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$L_k = \tau\{H_{k-1}-H_0,Q_{k-2}\} + \sum_{i=2}^{k}\frac{\tau^i}{i!}\underbrace{\{\ldots\{H_{k-i},Q_{k-i}\},Q_{k-i}\}\ldots\}}_{i \text{ times}} + H_k(u,v)$$

Then the formula for the required kth order asymptotic approximation of the transformed Hamiltonian may be written as

$$\tilde{H}_k = \tau\{H_0, Q_{k-1}\} + L_k \tag{11}$$

In this equation, called the homological equation, L_k is a known function of u and v, since, by (10), in order to determine it one has to know the lowest-order asymptotic approximations of H and Q, which have already been found. The functions in (11) that are unknown and must be determined are $\tilde{H}_k(u, v)$ and $Q_{k-1}(u, v)$.

To solve the homological equation for these functions, we observe that the Poisson bracket $\{H_0, Q_{k-1}\}$ is the total derivative with respect to t of the function $Q_{k-1}(u, v)$, evaluated along trajectories of the system with Hamiltonian H_0, that is, along a family of mappings $u \to u\,\exp(i\lambda t)$, $v \to v\,\exp(-i\lambda t)$

$$\{H_0, Q_{k-1}\} = dQ_{k-1}/dt.$$

However, by definition $\tilde{H}_k$ consists solely of invariants of the same linear system, and hence it follows from Eq. (11) that

$$\int_0^t L_k(ue^{i\lambda t},ve^{-i\lambda t})dt = t\tilde{H}_k - \tau Q_{k-1} + \tau Q_{k-1}(t) \tag{12}$$

This formula completes the algorithm for Birkhoff normalization: (a) calculate the function $L_k(u, v)$ and (b) evaluate the integral with respect to time of this function along trajectories of the generating system.

Then the coefficient of t is the required normal form, while the coefficient of τ, which is independent of t, is the required asymptotic approximation of the generating Hamiltonian, needed to search for the next approximation.

The first step of this recurrent scheme is to evaluate the integral of the first asymptotic approximation of the given Hamiltonian along the trajectories of the generating system

$$\int_0^t H_1(ue^{i\lambda t},ve^{-i\lambda t})dt = t\tilde{H}_1 - \tau Q_0 + \tau Q_0(t)$$

Example (Duffing's equation) $\ddot{q} + q + q^3 = 0$. The Hamiltonian corresponding to this equation, $H = (p^2 + q^2 + q^4/2)/2$, is reduced by the change of variables $x = q - ip, y = q + ip$ to the form $H = i[xy + x + y)^4/32]$. There are no third-degree terms in the Hamiltonian, and therefore the parameter

ε that distinguishes terms of different orders from each other may be defined subject to the condition $x = \sqrt{(\varepsilon)}x', y = \sqrt{(\varepsilon)}y'$. Hence the Hamiltonian may be written as follows (omitting primes):

$$H = i[xy + \varepsilon(x + y)^4/32]$$

First approximation. The transformation $x \to ue^{it}, y \to ve^{-it}, \varepsilon = \tau$ yields

$$\int_0^t H_1 dt = \int_0^t H dt = i\int_0^t \left[uv + \frac{\tau}{32}(ue^{it} + ve^{-it})^4 \right] dt =$$

$$= \tilde{H}_1 + \tau Q_0 + \frac{\tau}{32}\left(\frac{1}{4}u^4 e^{4it} + 2u^3 ve^{2it} - 2uv^3 e^{-2it} - \frac{1}{4}v^4 e^{-4it} \right)$$

$$\tilde{H}_1 = i\left(uv + \frac{3\tau}{16}u^2 v^2 \right), \quad Q_0 = -\frac{1}{32}\left(\frac{1}{4}u^4 + 2u^3 v - 2uv^3 - \frac{1}{4}v^4 \right)$$

Second approximation. The function L_2 is

$$L_2 = \tau\{H_1 - H_0, Q_0\} + \frac{\tau^2}{2}\{\{H_0, Q_0\}, Q_0\} + H_2$$

where

$$H_1 - H_0 = \frac{i\tau}{32}(u + v)^4, \quad \{H_0, Q_0\} = \frac{1}{\tau}(\tilde{H}_1 - H_1) = -\frac{i}{32}(u^4 + 4u^3 v + 4uv^3 + v^4), \quad H_2 = H.$$

Calculations analogous to those carried out in the first approximation give

$$\tilde{H}_2 = i\left(uv + \frac{3\tau}{16}u^2 v^2 - \frac{17}{256}\tau^2 u^3 v^3 \right)$$

(only the terms with the factor t are singled out). Since we are limiting ourselves to two approximations, there is no need to determine Q_1, which is needed to seek the third approximation.

To return to the initial scale of measurement of the variables, we only need to set $\tau = 1$.

Remarks. 1. To simplify the calculations with the recurrent formula (10), all terms involving the factor τ^m with $m > k$ should be equated to zero.

2. There is also a definition of BNF based on the use of "action-angle" variables [7] $p_i = \sqrt{(2\rho_i)} \cos \varphi_i, q_i = \sqrt{(2\rho_i)}$ sin φ_i. A BNF is a polynomial in powers of ρ and it exists only in non-resonant cases. In the case of resonance the BNF is said to be resonant and it now depends not only on the action variable but also on resonant combinations of angles.

The definition presented in this study is more convenient, for the following reasons. First, it takes care of both resonant and non-resonant cases. Second, it appeals not to external criteria, such as the form of the dependence on ρ and φ, but to notions that determine the fundamental essence of the normal form: the normal form may depend only on polynomial invariants of the generating system. It is quite unimportant whether the invariants themselves are associated with "identical" or "non-identical" resonances. Third, the definition of normal form for Hamiltonian systems should be a special case of the definition of normal form for arbitrary systems (Poincaré form). Fourth, a definition for invariant notions is preferably given in invariant form. In this investigation this condition is satisfied not only by the definition but also by the algorithm: when one is evaluating the integral on the left of (12) along trajectories of a degenerate system, it is immaterial in what variables these trajectories are described, that is, preliminary reduction of the Hamiltonian H_0 to canonical form is not necessary.

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